Deflections of Inelastic Beams With Nonuniform Temperature Distribution

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Let x and y be the coordinates of the beam cross section with origin at the centroid of the cross section. In terms of reference axes x_R and y_R,

$$x = x_R - \frac{\int E_R x_R dA}{\int E_R dA} \qquad y = y_R - \frac{\int E_R y_R dA}{\int E_R dA} \tag{1}$$

where E_R is the room-temperature modulus of elasticity. Let W be the deflection of the centroid in the direction y, z the length variable for the beam, α the coefficient of thermal expansion, and T the temperature change. Then

$$d^2W/dz^2 = M/(EI) = K/c \tag{2}$$

where

$$\left.\begin{aligned} K &= K_{ap} + K_T + K_p \\ K_{ap} &= cM_x \Big/ \int E_R y^2 dA \\ K_T &= c \int E_R \alpha T y dA \Big/ \int E_R y^2 dA \end{aligned}\right\} \tag{3}$$

with c the distance to the extreme fiber, M_x the applied moment, and K_p the rotation of the cross section due to inelastic effects. K_p can be determined by the procedures described in Ref. 1.

With K_p known only at certain cross sections, it is necessary to integrate Eq. (2) numerically. For a beam of length L divided into N segments, the cantilever deflection at segment i relative to the fixed end $z = 0$ can be approximated by

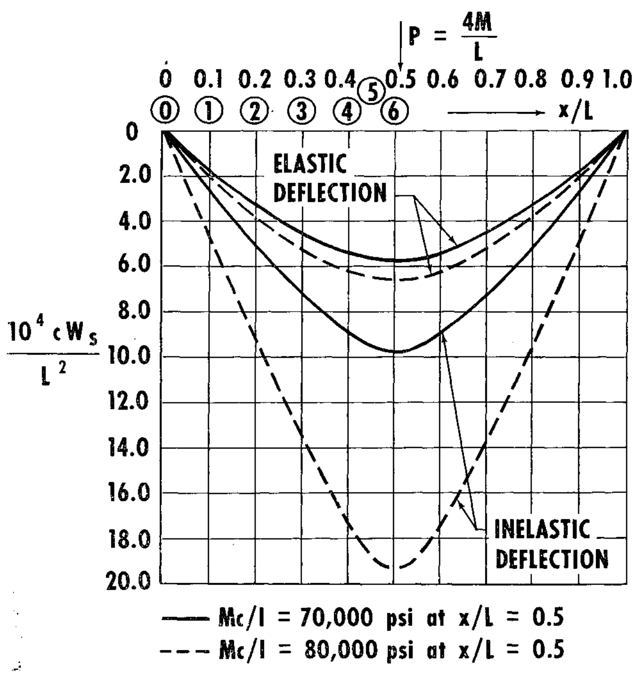

Fig. 1. Deflection of 2024-T3 rectangular bar.

$$\left.\begin{aligned} W_{ic} &= W_{i-1,c} + \Delta W_i + 2\Delta z_i \sum_{j=1}^{i-1} \frac{\Delta W_j}{\Delta z_j} \\ \Delta W_i &= [(K/c)_i + (K/c)_{i-1}](\Delta z_i/2)^2 \end{aligned}\right\} \tag{4}$$

The deflection of the simply supported beam is

$$W_{is} = \frac{W_{Nc}}{L} \sum_{j=1}^{i} \Delta z_j - W_{ic} \tag{5}$$

where W_{Nc} is the deflection at the free end of the cantilever beam.

Note that within the limitations on the Eq. (2), the cross section of the beam may be variable and M_x and T may be functions of z.

As an example, find the deflection of a simply supported aluminum-alloy rectangular bar loaded by a concentrated load at the center in such a way as to produce a maximum value for Mc/I of 70,000 psi. Take the bar at room temperature and use the load-strain curve for bending of rectangular bars given in Fig. 4 of Ref. 1. Divide the beam into segments as shown in Fig. 1 and take $K_{ap} + K_p = F_{apm}/E_R + e_{psm}$ in Fig. 4 of Ref. 1, with $E_R = 10^7$ psi. Using $K_i = (K_{ap} + K_p)_i = 0$, 0.0014, 0.0028, 0.0047, 0.0083, 0.0125, and 0.0210 for the six points, the inelastic deflections are calculated by Eqs. (4) and (5) and are shown in Fig. 1. The elastic deflections were calculated from $K_i = (K_{ap})_i = 0$, 0.0014, 0.0028, 0.0042, 0.0056, 0.0063, and 0.0070 and checked by direct integration of Eq. (2). Fig. 1 also shows the case of maximum Mc/I of 80,000 psi with $(K_{ap} + K_p)_i = 0$, 0.0016, 0.0032, 0.0057, 0.014, 0.030, and 0.050. Note that for this latter case, Fig. 1 shows a hinge action at the center of the beam where the inelastic effects are large, with practically a straight-line deflection on either side.

References

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Couette-Type Flow Through a Porous-Walled Annulus

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In a recent communication,[1] Lilley investigated the flow of an incompressible viscous fluid between two parallel and uniformly porous planes, one of which is fixed while the other is moving in its own plane with a uniform velocity in the main flow direction. The study revealed that for uniform injection of fluid, a smaller axial pressure gradient would provoke separation at the fixed wall than in the case of suction at the fixed wall. It was also shown that the shear stress experienced by the fixed wall is considerably reduced by blowing and increased by suction at the fixed wall. In this note, we study these features of porous-wall Couette-type flow through an annular tube; it is found that the results for annular flow are similar to those in Ref. 1. The existence of points of inflection in the velocity profile is also discussed.

Basic Equations and Solution

We consider steady-state laminar flow of an incompressible viscous fluid through an annulus whose inner tube ($r = b$) is stationary and whose outer tube ($r = a$) is moving with a uniform velocity U in the axial (x) direction. Assuming that there is no transverse component of velocity and also that the radial v_r and

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axial v_x components of velocity are functions of r alone, the equations of continuity and motion are, respectively,

$$(d/dr)(rv_r) = 0 \tag{1}$$

$$v_r \frac{d}{dr}(v_r) = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rv_r)\right] \tag{2}$$

$$v_r \frac{d}{dr}(v_x) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \left[\frac{d^2 v_x}{dr^2} + \frac{1}{r}\frac{dv_x}{dr}\right] \tag{3}$$

where p, ρ, and ν are pressure, density, and kinematic viscosity of the fluid.

Eq. (1), on integration, yields

$$v_r = \lambda v_0/\eta \tag{4}$$

in which $\lambda = b/a$ is annular radius ratio, $\eta = r/a$ is a nondimensional radial-distance parameter, and v_0 is the velocity with which the fluid is blown into the fixed tube ($\eta = \lambda$).

Eq. (4) also implies a porous moving tube ($\eta = 1$) through which the fluid flows out with a uniform velocity λv_0. Writing

$$\xi = x/a, \qquad \varpi = \frac{p}{{}^1/_2 \rho U^2}$$

$$Re = aU/\nu, \qquad V_x = v_x/U, \text{ etc.}$$

Eqs. (2) and (3) reduce to

$$(\partial\varpi/\partial\eta) - 2\lambda^2(V_0{}^2/\eta^3) = 0 \tag{5}$$

$$\eta \frac{d^2 V_x}{d\eta^2} + \frac{dV_x}{d\eta} - R\frac{dV_x}{d\eta} + c\eta = 0 \tag{6}$$

where $R = bv_0/\nu$ is cross-flow Reynolds number and $c = -(Re/2)(\partial\varpi/\partial\xi)$ is a term which refers to a Couette-type problem with superimposed axial pressure gradient. Integrating Eq. (5), we get

$$\varpi(\xi,\eta) + \lambda^2(V_0{}^2/\eta^2) + (2c/Re)\xi = 0 \tag{7}$$

whence

$$\varpi(0,\eta) - \varpi(\xi,\eta) = (2c/Re)\xi \tag{8}$$

which shows that pressure decreases linearly in the main flow direction. The solution of Eq. (6) subject to the boundary conditions

$$\left.\begin{matrix} V_x = 0, & \eta = \lambda \\ V_x = 1, & \eta = 1 \end{matrix}\right\} \tag{9}$$

is

$$V_x = \frac{\lambda^R}{\lambda^R - 1} - \frac{c(\lambda^2 - \lambda^R)}{2(R-2)(1-\lambda^R)} + \left[\frac{1}{1-\lambda^R} - \frac{c(1-\lambda^2)}{2(R-2)(1-\lambda^R)}\right]\eta^R + \frac{c}{2(R-2)}\eta^2 \tag{10}$$

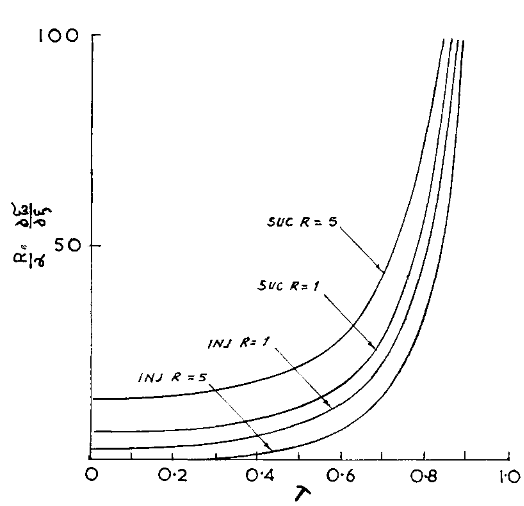

FIG. 1. Plot of $(Re/2)(\partial\omega/\partial\xi)$ against annulus radius ratio λ for injection and suction Reynolds number 1 and 5.

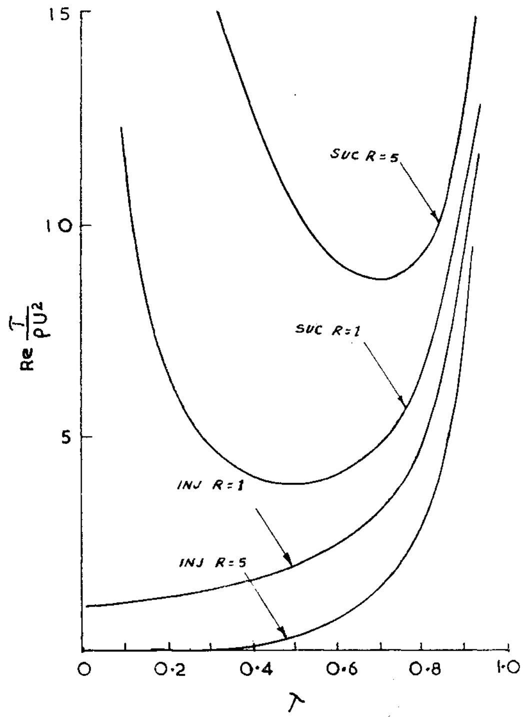

FIG. 2. Plot of $Re(\tau/\rho U^2)$ against annulus radius ratio λ for injection and suction Reynolds number 1 and 5.

where R is injection Reynolds number. In the case of suction of fluid through the fixed tube, R is replaced by $-R$ and the solution is given by

$$V_x = \frac{1}{1-\lambda^R} + \frac{c(1-\lambda^{R+2})}{2(R+2)(1-\lambda^R)} + \left[\frac{c\lambda^R(\lambda^2-1)}{2(R+2)(1-\lambda^R)} - \frac{\lambda^R}{1-\lambda^R}\right]\frac{1}{\eta^R} - \frac{c}{2(R+2)}\eta^2 \tag{11}$$

where R is suction Reynolds number.

The solution given by Eq. (10) does not hold for injection Reynolds number $R = 2$. This special case is dealt with later.

SEPARATION AND SKIN FRICTION AT THE FIXED TUBE

The separation at the fixed tube $\eta = \lambda$ is obtained when

$$(dV_x/d\eta)_{\eta=\lambda} = 0 \tag{12}$$

which, on using Eq. (10) for V_x, gives

$$c = \frac{2R(R-2)\lambda^{R-2}}{R(1-\lambda^2)\lambda^{R-2} - 2(1-\lambda^R)} \tag{13}$$

For suction Reynolds numbers, the expression for c is

$$c = \frac{2R(R+2)}{2(1-\lambda^R)\lambda^2 - R(1-\lambda^2)} \tag{14}$$

Eqs. (13) and (14) give the axial pressure gradient $\partial\varpi/\partial\xi$ which when imposed on the flow would cause separation at the fixed tube.

When the axial pressure gradient is zero ($c = 0$) the skin friction at the fixed tube is given by

$$\frac{1}{Re}\left(\frac{dV_x}{d\eta}\right)_{\eta=\lambda} = \frac{\tau}{\rho U^2} = \frac{R}{Re}\frac{\lambda^{R-1}}{1-\lambda^R} \tag{15}$$

where R is injection Reynolds number, and

$$\frac{\tau}{\rho U^2} = \frac{R}{Re}\frac{1}{\lambda(1-\lambda^R)} \tag{16}$$

where R is suction Reynolds number.

Thus we see from Eqs. (15) and (16) that skin friction is considerably more reduced when fluid is injected into the fixed tube then it is in the case of suction.

EXISTENCE OF POINTS OF INFLEXION

The existence of points of inflection in the velocity profile is of importance in flow stability considerations.[2] They are given by those values of η which satisfy the relations

$$d^2V_x/d\eta^2 = 0; \quad d^3V_x/d\eta^3 \neq 0 \tag{17}$$

Case 1. Injection Reynolds Number

Using Eq. (10) for V_x, the point of inflection in this case is given by

$$\eta = \left[\frac{2c(1-\lambda^R)}{R(R-1)\{c(1-\lambda^2) - 2(R-2)\}}\right]^{1/(R-2)} \tag{18}$$

The condition that it should lie in the flow region—i.e., $\lambda \leq \eta \leq 1$—for $R > 2$ is that

$$\frac{2R(R-1)(R-2)\lambda^{R-2}}{2(1-\lambda^R) - R(R-1)(\lambda^{R-2} - \lambda^R)} \leq \frac{Re}{2}\frac{\partial\varpi}{\partial\xi} \leq \frac{2R(R-1)(R-2)}{2(1-\lambda^R) - R(R-1)(1-\lambda^2)} \tag{19}$$

For injection Reynolds numbers less than 2, the signs of inequality are altered in Eq. (19).

Case 2. Suction Reynolds Number

Using Eq. (11), the point of inflection in the velocity profile is given in this case by

$$\eta = \left[\frac{R(R+1)\{c\lambda^R(\lambda^2-1) - 2(R+2)\lambda^R\}}{2c(1-\lambda^R)}\right]^{1/(R+2)} \tag{20}$$

and the limits on the pressure gradient so that η [given by Eq. (20)] should lie in the flow region are given by the inequality

$$\frac{2R(R+1)(R+2)}{2\lambda^2(1-\lambda^R) + R(R+1)(1-\lambda^2)} \geq \frac{Re}{2}\frac{\partial\varpi}{\partial\xi} \geq \frac{2R(R+1)(R+2)\lambda^R}{2(1-\lambda^R) + R(R+1)(1-\lambda^2)\lambda^R} \tag{21}$$

SPECIAL CASE—INJECTION REYNOLDS NUMBER $Re = 2$

In this case, the solution of Eq. (6) subject to the boundary conditions (9) is

$$V_x = \left(\frac{c}{2}\ln\lambda - 1\right)\frac{\lambda^2}{1-\lambda^2} + \frac{[1-(c/2)\lambda^2\ln\lambda]}{1-\lambda^2}\eta^2 - \frac{c}{2}\eta^2\ln\eta \tag{22}$$

The value of c for which separation occurs at the fixed tube is

$$c = 4/(1-\lambda^2 + 2\ln\lambda) \tag{23}$$

whence

$$\partial\varpi/\partial\xi = -8/[Re(1-\lambda^2+2\ln\lambda)] \tag{24}$$

The skin friction at the fixed tube is given by

$$\tau/(\rho U^2) = (1/Re)[2\lambda/(1-\lambda^2)] \tag{25}$$

Fig. 1 shows that for a fixed Re, the axial pressure gradient which would provoke separation at the fixed tube increases asymptotically as $\lambda \to 1$ for both suction and injection, and also that it is less in the case of injection than that of suction. Fig. 2 shows that skin friction (for a fixed Re) is less in the case of injection than that of suction and also that for injection it increases asymptotically as $\lambda \to 1$, and for suction it increases asymptotically for annulus radius ratio tending to both 0 and 1.

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The Characteristic Form of the Equations of One-Dimensional Magnetohydrodynamic Flow With Oblique Magnetic Field

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THIS NOTE presents a derivation of the characteristic form of the equations governing the one-dimensional unsteady flow of an ideal, inviscid, perfectly conducting, compressible fluid subjected to an oblique magnetic field. It is assumed that all dependent variables are functions of only one space variable x and the time t, i.e., it is the projections of the characteristics (really, bicharacteristics) from (x,y,t) space onto the (x,t) plane that are considered. The resultant characteristic system, which contains the characteristic systems for a purely transverse field[4–6] and the nonmagnetic case[1] as special cases, is suitable for approximate treatment by finite-difference techniques.

The basic equations are:[3]

$$c_t + uc_x + (\gamma - 1)cu_x/2 = 0 \tag{1}$$

$$u_t + uu_x + 2cc_x/(\gamma-1) + b_2{}^2B_{2x}/B_2 - c^2s_x/\gamma(\gamma-1)c_v = 0 \tag{2}$$

$$v_t + uv_x - b_1{}^2B_{2x}/B_1 = 0 \tag{3}$$

$$B_{2t} - B_1v_x + u_xB_2 + uB_{2x} = 0 \tag{4}$$

$$s_t + us_x = 0 \tag{5}$$

where $\mathbf{q} = (u,v,0)$, $\mathbf{B} = (B_1, B_2, 0)$, c, s, $b_i{}^2 = B_i{}^2/\mu\rho(i = 1,2)$, ρ, μ and γ are, respectively, the particle velocity, induction, local speed of sound, specific entropy, square of the Alfvén speed, density, permeability, and ratio of specific heats at constant pressure c_p and at constant volume c_v. Partial derivatives are denoted by subscripts, and all dependent variables are functions of x and t alone. As a consequence of Maxwell's equations, there is the further condition that B_1 be constant.

To determine the characteristic curves of the system (1)–(5), it is convenient to introduce[2] new independent variables $\phi(x,t)$, $\psi(x,t)$. The possible characteristic manifolds are given by solutions of the first-order partial differential equation for ϕ:

$$[\phi_t + u\phi_x][\phi_t{}^4 + 4u\phi_x\phi_t{}^3 + (6u^2 - b_1{}^2 - b_2{}^2 - c^2)\phi_x{}^2\phi_t{}^2 + \{4u^3 - 2u(b_1{}^2 + b_2{}^2 + c^2)\}\phi_x{}^3\phi_t + \phi_x{}^4\{u^2(u^2 - b_1{}^2 - b_2{}^2) - c^2(u^2 - b_1{}^2)\}] = 0 \tag{6}$$

The first factor shows that one characteristic curve is given by $dx/dt = u$. If $\phi(x,t) =$ constant is characteristic, then it follows that $dx/dt = -\phi_t/\phi_x$, so that the introduction of $\lambda = dx/dt$ in the remaining factor of (6) leads to a fourth-order algebraic equation for λ. This takes a better form if the substitution $\lambda = u + a$ is made; the resultant algebraic equation for a is

$$a^4 - \omega^2a^2 + b_1{}^2c^2 = 0 \tag{7}$$

where $\omega^2 = c^2 + b_1{}^2 + b_2{}^2$. The larger and the smaller of the roots a > 0 of (7) will be denoted by a_f (fast speed) and a_s (slow speed), respectively.[3]

Thus, introducing characteristic parameters $(\alpha, \beta, \xi, \eta, \zeta)$, the first five equations of the characteristic system are

$$\left.\begin{array}{l} x_\beta = (u + a_f)t_\beta,\ x_\alpha = (u - a_f)t_\alpha,\ x_\xi = ut_\xi \\ x_\eta = (u + a_s)t_\eta,\ x_\zeta = (u - a_s)t_\zeta \end{array}\right\} \tag{8}$$

To determine the remaining equations of the characteristic system, let the equations (1)–(5) be multiplied by ν_1, ν_2, ν_5, ν_3, ν_4, respectively, and add the resultant equations:

$$\left.\begin{array}{l} [\nu_1(\gamma - 1)c/2 + \nu_2u + \nu_3B_2]u_x + \nu_2u_t + \\ [\nu_1u + 2c\nu_2/(\gamma-1)]c_x + \nu_1c_t + [\nu_5u - \nu_3B_1]v_x + \\ \nu_5v_t + [\nu_2b_2{}^2/B_2 - \nu_5b_1{}^2/B_1 + \nu_3u]B_x + \\ \nu_3B_t + [\nu_4u - \nu_2c^2/\gamma(\gamma-1)c_v] + \nu_4s_t = 0 \end{array}\right\} \tag{9}$$